ON THE MOTION OF COMPRESSIBLE FLUIDS IN PLANE CHANNELS WITH MOVING WALLS

(O DVIZHENII SZHIMAEMOI ZHIDKOSTI V PLOSKIKH KANALAKH S PODVIZHNYMI STENKAMI)

PMM Vol.27, No.4, 1963, pp. 719-725

A. F. SIDOROV
(Cheliabinsk)

(Received April 12, 1963)

Flows of the double-wave type in two and three dimensions for polytropic gases have been studied in [1-6]. In those papers, on the basis of properties of potential flows, equations were derived describing motions of the double-wave type, and many applications of the theory of these flows to concrete problems in gas dynamics were considered.

The flows considered in [1] may be used to solve problems on the steady flows past some special surfaces with the unperturbed flow supersonic. In [5] and [6] the theory of plane double waves was applied to construct flows behind asymmetric shock and detonation waves of constant strength.

Below we shall consider a new application of the theory of plane double waves, again under the assumption of potential flow. It will be shown that in the class of double waves, it is possible to have a steady plane isothermal or polytropic gas flow adjoin an unsteady plane double-wave flow across a stationary characteristic. This property permits us, under the assumption of the hyperbolicity of the system of equations being considered (we consider supersonic flows), to formulate many boundary value problems in the hodograph plane for the sound velocity $C(u_1, u_2)$ and the potential $\Phi^{\circ}(u_1, u_2)$ (u_1 and u_2 being the components of the vector $\mathbf{u}$).

The formulation of the problems is in some sense similar to the basic boundary value problems for plane steady potential flows in curved channels [9]. If in steady flow the sound speed is found from Bernoulli's equation, then in this case instead of Bernoulli's equation we must consider a nonlinear second order equation for the sound speed

$C(u_1, u_2)$ in the hodograph plane, known from the theory of double waves (see [3,4]); for this equation, we must solve either a Goursat problem or a mixed problem.

Moreover, in this fundamental case, the velocity distribution along the moving wall and along the line separating the steady and unsteady flows maps into a time-independent curve $K(u_1, u_2) = 0$ in the hodograph plane.

The formulated boundary value problems permit in principle the solution of gas motions in curved channels, whose walls are stationary up to a certain point and from there on move according to a definite rule, so that the flow in the region of the physical plane bounded by the fixed characteristics through the last fixed point on the channel walls is steady and the flow behind those characteristics is unsteady.

As an illustration, we use Masseau's method of characteristics to solve for an isothermal gas the problem of the joining of a steady simple wave to an unsteady double wave across a stationary characteristic.

1. The system of equations describing double waves under the assumption of straight-line generatrices is (see [4])

$$\tfrac{1}{2}(\gamma-1)\theta\,[(1-\theta_1^2)\theta_{22}+2\theta_1\theta_2\theta_{12}+(1-\theta_2^2)\theta_{11}]+\tfrac{1}{2}(\gamma-3)(\theta_1^2+\theta_2^2)+2=0 \tag{1.1}$$

$$(1-\theta_1^2)\,\Phi_{22}^\circ+2\theta_1\theta_2\Phi_{12}^\circ+(1-\theta_2^2)\,\Phi_{11}^\circ=0 \tag{1.2}$$

$$x_i=[u_i+\tfrac{1}{2}(\gamma-1)\theta\theta_i]\,t+\Phi_i^\circ \qquad (i=1,2) \tag{1.3}$$

Here

$$\theta_i=\frac{\partial\theta}{\partial u_i},\quad \Phi_i^\circ=\frac{\partial\Phi^\circ}{\partial u_i},\quad \theta_{ik}=\frac{\partial^2\theta}{\partial u_i\,\partial u_k},\quad \Phi_{ik}^\circ=\frac{\partial^2\Phi^\circ}{\partial u_i\,\partial u_k} \qquad (i,\ k=1,2)$$

We assume isentropic flows; the equation of state is in the form

$$p=a^2(S)\,\rho^\gamma \qquad \left(\theta=\frac{2}{\gamma-1}\,C\right)$$

Here p is the pressure, S the entropy, γ the adiabatic exponent, and ρ the density. We assume straight-line generatrices; this means that the velocity vector $\mathbf{u}$ is constant along straight lines in the x_1, x_2, t space, given by relation (1.3). After solving the system (1.1) and (1.2) for the functions θ and Φ°, defined in the hodograph plane, we find the flow in the physical plane from relation (1.3). We note that the types of equation (1.1) for θ and of equation (1.2) for Φ° are the same, and are determined by the sign of the expression

$$K = \theta_1^2 + \theta_2^2 - 1 \qquad (1.4)$$

In what follows, we shall assume that $K > 0$, i.e. that equations (1.1) and (1.2) are hyperbolic and the characteristics are real for both equations. Let us consider the following problem.*

Is it possible to join an unsteady double-wave flow to a steady plane potential flow? In plane steady potential flow, we have the Bernoulli integral. We write this as

$$\tfrac{1}{4}(\gamma - 1)\theta^2 + \tfrac{1}{2}(u_1^2 + u_2^2) = A^2 = \text{const} \qquad (1.5)$$

Let us assume that the unsteady double-wave flow is bounded from a two-dimensional steady flow by some line $\varphi(u_1, u_2) = 0$ in the hodograph plane. Then along this line, the function θ defined by equation (1.1) must satisfy the following condition, derived from (1.5) and (1.3)

$$\tfrac{1}{2}(\gamma - 1)\theta\theta_i + u_i = 0 \qquad (i = 1, 2) \qquad (1.6)$$

(the derivatives θ_i are continuous across the line in question).

From (1.3) and (1.6), it follows that the boundary between the flows considered may only be a stationary curve, defined by

$$x_i = \Phi_i^{\circ} \qquad (i = 1, 2), \qquad \varphi(u_1, u_2) = 0 \qquad (1.7)$$

If the curve $\varphi(u_1, u_2) = 0$ degenerates into the point $u_1 = c_1 = \text{const}$, $u_2 = c_2 = \text{const}$ with $c_1^2 + c_2^2 > 0$, then there is no common line between the flows in the physical plane. An exception is the case $c_1 = c_2 = 0$, for which it was shown [6] that to a region of rest may be adjoined an unsteady double wave across a stationary weak discontinuity. This case will not be considered in the present work.

Relation (1.6) may be interpreted as a set of initial Cauchy data for equation (1.1) along the line $\varphi(u_1, u_2) = 0$. Since equation (1.1) is satisfied by the function θ, determined from Bernoulli's integral (1.5), it is necessary to consider the line $\varphi(u_1, u_2) = 0$ as a

---

* We note in passing that M. Burnat has constructed in [7] examples of some constant plane flows adjoined to unsteady flows, but only in the class of simple waves. We shall consider the possibility of constructing unsteady flows of a much wider class, as we shall see below.

characteristic of the equations (1.1) and (1.2) in order to obtain an unsteady flow. We write the equations of the characteristic strips (see [8, Chap.3]) for (1.1) and (1.2) in the form

$$(1-\theta_2^2)\,du_2^2 - 2\theta_1\theta_2 du_2 du_1 + (1-\theta_1^2)\,du_1^2 = 0 \tag{1.8}$$

$$\tfrac{1}{2}(\gamma-1)\theta\,(1-\theta_2^2)\,d\theta_1 du_2 + [\tfrac{1}{2}(\gamma-3)(\theta_1^2+\theta_2^2)+2]\,du_1du_2 + $$
$$+\tfrac{1}{2}(\gamma-1)\theta\,(1-\theta_1^2)\,du_1 d\theta_2 = 0$$

$$(1-\theta_2^2)\,du_2 d\Phi_1^\circ + (1-\theta_1^2)\,du_1 d\Phi_2^\circ = 0$$

$$d\theta = \theta_1 du_1 + \theta_2 du_2, \qquad d\Phi^\circ = \Phi_1^\circ du_1 + \Phi_2^\circ du_2$$

The characteristics in the hodograph plane, defined by the first equation in (1.8) with θ substituted from the Bernoulli integral (1.5), coincide with the corresponding characteristics of steady flow and consequently will be epicycloids (see [9]).

Since the coefficients for the highest derivatives for Φ° and θ along the characteristics in question coincide with the corresponding coefficients of the equation for the potential in steady flow, and since Bernoulli's integral (1.5) satisfies equation (1.1), the condition for the characteristic strips (1.8) is exactly satisfied along the stationary characteristics considered. Thus, the assertion that unsteady double waves may adjoin steady potential plane flows is proved.

An entirely similar situation obtains for an isothermal gas, in which case instead of equations (1.1) to (1.3) we have

$$(1-q_1^2)(q_{22}+1) + 2q_1q_2q_{12} + (1-q_2^2)(q_{11}+1) = 0 \tag{1.9}$$

$$(1-q_1^2)\,\Phi_{22}^\circ + 2q_1q_2\Phi_{12}^\circ + (1-q_2^2)\,\Phi_{11}^\circ = 0 \tag{1.10}$$

$$x_i = (u_i + q_i)\,t + \Phi_i^\circ \qquad (i = 1, 2) \tag{1.11}$$

where

$$q = \ln\rho, \qquad q_i = \frac{\partial q}{\partial u_i}, \qquad q_{ik} = \frac{\partial^2 q}{\partial u_i\,\partial u_k}$$

The characteristics for steady flow in the hodograph plane are evolutes of the circle; thus in this case the characteristic strips for equation (1.10) may be written in parametric form as:

$$u_1 = \cos p + (p-p_0)\sin p, \qquad q_1 = -\cos p - (p-p_0)\sin p$$
$$u_2 = \sin p - (p-p_0)\cos p, \qquad q_2 = -\sin p + (p-p_0)\cos p$$
$$q = c - \tfrac{1}{2}(p-p_0)^2 \qquad (c = \text{const}, \ p - \text{parameter})$$

2. Let us consider some properties of the moving walls, which are consistent with the class of flows being studied. By a moving wall in this connection we mean some curve in the x_1, x_2 plane moving with time and given by $\psi(x_1, x_2, t) = 0$, across which there is no flow. Along the wall curve, the following kinematic boundary condition must hold: the normal component of the fluid velocity to the wall must equal the normal wall velocity.

Let $\mathbf{n}$ be the normal to the wall curve at the point x_1, x_2 at the instant t, and let D be the normal velocity of the wall at the same point. The kinematic boundary condition is then

$$\mathbf{n}\cdot\mathbf{u} = D \tag{2.1}$$

From (2.1) we obtain an equation for the function ψ

$$\partial\psi / \partial x_1 u_1 + \partial\psi / \partial x_2 u_2 + \partial\psi / \partial t = 0 \tag{2.2}$$

Substituting into (2.2) the functions u_i, found from relation (1.3) (after solving equations (1.1) and (1.2)), and solving for (2.2) the Cauchy problem $\psi = \psi(x_1, x_2)$ at $t = t_0$, we may find for a given flow different moving walls of shapes known at the instant $t = t_0$.

We note one further property. We shall consider the case when the motion of the wall corresponds to a fixed curve in the hodograph plane.

Theorem. In the class of unsteady plane flows of the double-wave type, there exists no stationary curved wall with a time-independent velocity distribution $K(u_1, u_2) = 0$.

In fact, assume that such a wall is found and the curve $K(u_1, u_2) = 0$ corresponds to some cylindrical surface in the x_1, x_2, t space. Then from the presence of the straight-line generators belonging to this surface, along which u_1 and u_2 are constant, it follows that the normals to the wall surface are constant, and consequently the wall is straight.

Let us find D and $\mathbf{n}$, if the motion of the curve in the x_1, x_2 plane is defined by (1.3) and the shape of this curve in the hodograph plane is known and given by the equation

$$u_2 = f(u_1) \tag{2.3}$$

Setting

$$\Delta_i = u_i + {}^1/_2 (\gamma - 1)\theta\theta_i \qquad (i = 1, 2) \tag{2.4}$$

we have from (1.3)

$$dx_i = \Delta_i' t du_1 + \Delta_i dt + \Phi_i^{\circ\prime} du_2 \qquad (i = 1, 2) \tag{2.5}$$

where the prime indicates differentiation along the curve (2.3). For a vector perpendicular to the curve in question at the instant t at the point u_1, u_2, we have

$$\frac{dx_1}{dx_2} = -\frac{\Delta_2' t + \Phi_2^{\circ\prime}}{\Delta_1' t + \Phi_1^{\circ\prime}} \tag{2.6}$$

Solving the system of equations (2.5) and (2.6), we find

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{(\Delta_2' t + \Phi_2^{\circ\prime})\,[\Delta_1(\Delta_2' t + \Phi_2^{\circ\prime}) - \Delta_2(\Delta_1' t + \Phi_1^{\circ\prime})]}{(\Delta_1' t + \Phi_1^{\circ\prime})^2 + (\Delta_2' t + \Phi_2^{\circ\prime})^2} \\ \frac{dx_2}{dt} &= \frac{-(\Delta_1' t + \Phi_1^{\circ\prime})\,[\Delta_1(\Delta_2' t + \Phi_2^{\circ\prime}) - \Delta_2(\Delta_1' t + \Phi_1^{\circ\prime})]}{(\Delta_1' t + \Phi_1^{\circ\prime})^2 + (\Delta_2' t + \Phi_2^{\circ\prime})^2} \end{aligned} \tag{2.7}$$

Taking into account the fact that

$$D = \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2\right]^{1/2} \tag{2.8}$$

we may write condition (2.1) in the form

$$\Delta_1(\Delta_2' t + \Phi_2^{\circ\prime}) - \Delta_2(\Delta_1' t + \Phi_1^{\circ\prime}) = u_1(\Delta_2' t + \Phi_2^{\circ\prime}) - u_2(\Delta_1' t + \Phi_1^{\circ\prime}) \tag{2.9}$$

and consequently, since t is arbitrary, the following relations must hold along the moving wall curve:

$$\theta_1\Delta_2' - \theta_2\Delta_1' = 0, \qquad \theta_1\Phi_2^{\circ\prime} - \theta_2\Phi_1^{\circ\prime} = 0 \tag{2.10}$$

The first condition in (2.10) defines some relation $F(u_1, u_2, \theta_1, \theta_2, \theta) = c = \text{const}$ along the curve (2.3) for equation (1.1), in which the constant c must be found from the matching condition at the point where the stationary wall adjoins the moving wall. In exactly the same way, after the function $\theta(u_1, u_2)$ is found, condition (2.10) defines along (2.3) some relation $\Psi(u_1, u_2, \Phi_1^{\circ}, \Phi_2^{\circ}) = \text{const}$ for equation (1.2). We note that in steady flow, the first equation in (2.10) is automatically satisfied (since $\Delta_i = 0$), while the second equation in (2.10) expressed the fact that the normal fluid velocity at a stationary wall must vanish.

In the formulation of the problem considered, the velocity distribution is given along the moving wall curve, while the motion and shape of the wall is determined afterwards from equation (1.3). In principle, the other approach to the problem is also possible, in which the shape of the moving wall is given at some instant t. This leads to a condition of the type (2.10) in the physical plane; however, the curve in the hodograph plane corresponding to the moving wall (exactly as in the case of steady flow) is unknown and this complicates the solution of the problem, since equation (1.1) for θ is given in the hodograph plane.

The steady flow solution exactly satisfies the first condition (2.10);

consequently, an immediate joining (across a stationary characteristic) of a steady flow to an unsteady flow with moving walls of the type considered is not possible. For otherwise, from the solution of the mixed problem for equations (1.1) and (1.2) with initial data on the characteristic and on the noncharacteristic curve (2.3), we will only find a steady flow.

Thus, the general form of the flow in the hodograph plane may be represented as shown in Fig. 1.

The lines *AR* and *BR* correspond to stationary characteristics and bound the region *ARB*, in which the flow is steady. The regions *AECR* and *BRDF* correspond to unsteady flows, but in these regions there are no moving walls of the kind discussed. We may find the flows in these regions, given, for example, some relations on the functions θ_1, θ_2 and Φ_1°, Φ_2°, on the noncharacteristic curves *AC* and *BD*.

After solving the mixed problem in the regions *ACR* and *BRD*, we solve the Cauchy problem in the regions *AEC* and *BDF*, bounded by the characteristics *AE*, *EC* and *BF*, *FD* respectively, with initial data given on the lines *AC* and *BD*. In the region *RCOD* we solve the problem with data on the two characteristics *RC* and *RD*.

Finally, in the regions *APE* and *BQF* (lines *AP* and *BQ* corresponding to moving walls) we solve mixed problems with data given on the characteristics *AE* and *BF* and condition (2.10) given on the wall curves *AP* and *BQ*.

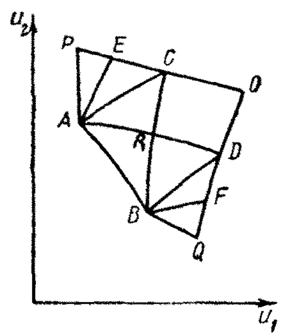

Fig. 1.

Each of the separate problems described may be solved for equations (1.1) and (1.2) numerically using the method of characteristics of Masseau; in addition, the derivatives in (2.10) are replaced by finite difference quotients and the problem is readily solved in the usual manner.

The approach discussed for solving the problem of joining a steady flow to an unsteady flow in a channel with moving walls is, evidently, not a unique one. In the given approach there are the following possibilities: we may arbitrarily give the shape of the lines *AC* and *BD* in the physical plane and in the hodograph plane, a combination of the functions θ, θ_1, θ_2 along them, and also the velocity distribution along the moving wall. This arbitrariness permits, in particular, the construction of unsteady flows with given properties (e.g. maximum acceleration of the steady flow at the beginning in regions *ACR* and *BRD* and then determination of the corresponding moving wall). In principle, we may give some additional conditions on the wall curves AP and BQ, solve the Cauchy problem in the regions APE and BFQ, and find the characteristics AE and BF, and then solve the problem with data on two characteristics in the regions $AECR$ and $BRDF$.

3. As an illustration of the given method, we carry out a calculation for an isothermal gas; for simplicity, we do not consider an unsteady double wave adjoining an arbitrary steady flow, but only one adjoining a steady simple wave, the boundary between them being a characteristic of one family only. Assume a steady uniform supersonic flow to occupy the inlet of a straight-walled channel, and then let one of the walls become curved, so that the uniform flow becomes a simple wave of the following form (Fig. 2):

$$u_1 = \cos s + (s + 1 - {}^1/_4\pi) \sin s, \qquad u_2 = \sin s - (s + 1 - {}^1/_4\pi) \cos s$$
$$s = x_1 + x_2 \tan s + {}^1/_4\pi, \qquad q = c - {}^1/_2 u_1^2 - {}^1/_2 u_1^2 \qquad (c = \text{const}) \qquad (3.1)$$

The curved part of the stationary wall OA, starting from the origin (0, 0), is given parametrically thus

$$x_1 = s - x_2 \tan s - {}^1/_4\pi, \qquad x_2 = \cos^2 s + \cos s \qquad (3.2)$$
$$\exp\left[\frac{s^2}{2} + \left(1 - \frac{\pi}{4}\right)s\right] \left(2 \int_{^1/_4\pi}^{s} \sin \exp\left[-\frac{s^2}{2} - \left(1 - \frac{\pi}{4}\right)s\right] ds - \frac{\sqrt{2}}{2} \exp\left[\frac{\pi^2}{32} - \frac{\pi}{4}\right]\right)$$

Up to the characteristic OB

$$x_1 + x_2 = 0 \qquad (3.3)$$

One channel wall is straight, and the flow is uniform with $u_2 = 0$ and $u_1 = \sqrt{2}$. The simple wave occurs in the region OAB. The shape of the steady characteristic AB is found after integrating the linear equation

$$\frac{dx_2}{ds} = \frac{(\cos^2 s - x_2)\,[u_1 + u_2 (s + 1 - {}^1/_4\pi)]}{2 \cos s\,(s + 1 - {}^1/_4\pi)} \qquad (3.4)$$

with initial condition at the point A, $s \in [1.26;\ 1/4\ \pi]$. In the region BAC the method of characteristics of Masseau is used to solve equations (1.10) and (1.11) for the unsteady double wave, in which along the noncharacteristic curve BC ($x_2 = -0.1593$) are given the conditions

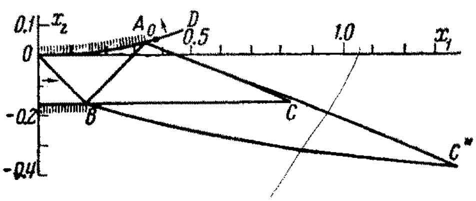

Fig. 2.

$$q_1 = \text{const} = -\sqrt{2}$$
$$\Phi_2^\circ = \text{const} = -0.1593,\ u_2 = 0 \qquad (3.5)$$

The solution and the characteristics in the hodograph plane are shown in Fig. 3.

The characteristic $A'B'$ corresponds to the simple wave region OAB in the physical plane, and the curve $A'D'$ to the moving wall AD. In the region $C'A'D'$ the solution is completely undetermined, and the position of the moving wall is considered only in the neighborhood of the point A. The shape of the moving wall is shown in Fig. 2. The normal velocity of the moving wall at the point o is $D = 0.103$, the point o corresponding the the point o' in the hodograph plane.

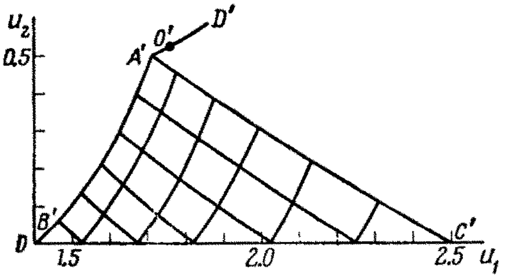

Fig. 3.

The region BAC in the physical plane corresponds to the region $B'A'C'$ in the hodograph plane at the instant $t = 0$, and the region BAC'' to $B'A'C'$ at $t = 0.5$. In the given case of an isothermal gas, the Jacobian of the coordinate transformation from the hodograph plane to the physical plane is

$$J = \begin{vmatrix} \partial x_1 / \partial u_1 & \partial x_2 / \partial u_2 \\ \partial x_2 / \partial u_1 & \partial x_2 / \partial u_2 \end{vmatrix} = [(1 + q_{11})(1 + q_{22}) - q_{12}^2]\, t^2 + \\ + [(1 + q_{11})\,\Phi_{22}^\circ + (1 + q_{22})\,\Phi_{11}^\circ - 2q_{12}\Phi_{12}^\circ]\, t + \Phi_{11}^\circ\Phi_{22}^\circ - \Phi_{12}^{\circ 2} \qquad (3.6)$$

At $t = 0$, it is immediately verified that

$$\Phi_{11}^\circ\Phi_{22}^\circ - \Phi_{12}^{\circ 2} \neq 0 \qquad (3.7)$$

i.e. mapping to the physical plane is possible, and no limiting line will appear for some time interval $t_0 > t > 0$. We remark here that the formulation of the problem does not permit (among double waves) the complete solution for unsteady flows joined to steady flows in moving-wall channels arising from a wall starting to move from some point on at some instant. This difficulty is connected with the appearance of limiting lines, if we consider the time $t < 0$. We shall not carry out here the investigation for the time of existence of the flows discussed.

In conclusion, the author thanks N.N. Ianenko for his attention and advice.

BIBLIOGRAPHY

1. Nikol'skii, A.A., O klasse adiabaticheskikh techenii gaza, kotorye v prostranstve godografa skorosti izobrazhaiutsia poverkhnostiami (On a class of adiabatic gas flows, which are represented by surfaces in the hodograph plane). *Collection of theoretical papers in aerodynamics*. Oborongiz, 1957.

2. Ryzhov, O.S., O techeniiakh s vyrozhdennym godografom (On flows with degenerate hodographs). *PMM* Vol. 24, No. 3, 1960.

3. Pogodin, Iu.Ia., Suchkov, V.A. and Ianenko, N.N., O begushchikh volnakh uravnenii gazovoi dinamiki (On propagating waves of the equations of gas dynamics). *Dokl. Akad. Nauk SSSR*, 119, No. 3, 1958.

4. Sidorov, A.F. and Ianenko, N.N., K voprosu o nestatsionarnykh ploskikh techeniiakh politropnovo gaza s priamolineinymi kharakteristikami (On the question of unsteady flows of polytropic gases having straight characteristics). *Dokl. Akad. Nauk SSSR*, 123, No. 5, 1958.

5. Sidorov, A.F., K voprosu ob udarnykh volnakh v techeniiakh politropnovo gaza, imeiushchikh priamolineinye kharakteristiki (On the question of shock waves in polytropic gas flows having straight characteristics). *PMM* Vol. 25, No. 2, 1961.

6. Sidorov, A.F., Nekotorye tochnye resheniia nestatsionarnoi dvumernoi gazovoi dinamiki (Some exact solutions of unsteady two-dimensional gas dynamics). *PMM* Vol. 24, No. 2, 1962.

7. Burnat, M., Zadacha Koshi dlia szhimaemykh techenii tipa prostykh voln (Cauchy problem for mixed flows of the simple-wave type). *Archiwum Mechaniki Stosowaney*, 3/4, 14, 1962.

8. Smirnov, V.I., *Kurs vysshei matematiki (Course in higher mathematics)*, Vol. 4. GITTL, 1953.

9. Von Mises, R., *Mathematical theory of compressible fluid flows*. Academic Press, 1958.

Translated by C.K.C.